ME 4555 - Lecture 28 - Frequency response

Instead of examining a system's response to a particular input (e.g. step or impulse), we can look at how a system responds to inputs at a particular frequency ω (in rad/sec).

> Key property of stable LTI systems: sinusoidal inputs produce sinusoidal outputs with the same frequency (but possibly different amplitude and phase) (once the transients die out).

Let's prove it! Suppose $G(s) = \dfrac{b(s)}{a(s)} = \dfrac{k(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$

Suppose our input is a sin + cos complex mixture:

$$u(t) = \cos\omega t + j\sin\omega t = e^{j\omega t}. \quad \text{So } U(s) = \frac{1}{s-j\omega}$$

We will compute $y(t)$ and then the responses to $\cos\omega t$ or $\sin\omega t$ will simply be $\mathrm{Re}(y(t))$ and $\mathrm{Im}(y(t))$, respectively.

$$Y(s) = G(s)U(s) = \frac{k(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} \cdot \frac{1}{s-j\omega}$$

PFE

$$= \frac{k_0}{s-j\omega} + \sum_{i=1}^{n} \frac{k_i}{s-p_i}$$

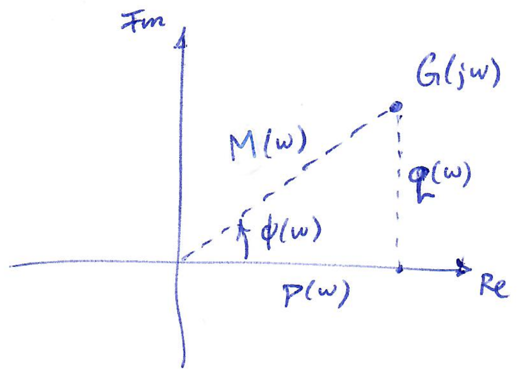
Using cover-up method, $k_0 = G(j\omega)$. Note: $G(j\omega)$ is just a complex number!

Therefore, $y(t) = \mathcal{L}^{-1}\left\{\dfrac{G(j\omega)}{s-j\omega}\right\} + \underbrace{\mathcal{L}^{-1}\left\{\sum_{i=1}^{n}\dfrac{k_i}{s-p_i}\right\}}$

These terms involve $e^{p_i t} \to 0$ as $t \to \infty$ since $\mathrm{Re}(p_i) < 0$ by assumption (stability!)

Now $\mathcal{L}^{-1}\left\{\frac{G(j\omega)}{s-j\omega}\right\} = G(j\omega)\,e^{j\omega t}$ since $G(j\omega)$ is a constant.

Since $G(j\omega)$ is an ordinary complex number, it has a magnitude + phase:



We can write: $G(j\omega) = p(\omega) + j\,q(\omega)$

or $G(j\omega) = M(\omega)\,e^{j\phi(\omega)}$

(cartesian or polar representation).

$M(\omega) = |G(j\omega)| = \sqrt{p(\omega)^2 + q(\omega)^2}$ is the <u>magnitude</u> of $G(j\omega)$.

$\phi(\omega) = \angle G(j\omega) = \arctan\left(\frac{q(\omega)}{p(\omega)}\right)$ is the <u>phase</u> of $G(j\omega)$.

Back to our system... we had $y(t) = G(j\omega)\,e^{j\omega t}$.

Using the polar form, we get:

$$y(t) = M(\omega)\,e^{j(\omega t + \phi(\omega))} = M(\omega)\cos(\omega t + \phi(\omega)) + j\,M(\omega)\sin(\omega t + \phi(\omega)).$$

But remember $u(t) = \cos(\omega t) + j\sin(\omega t)$

Therefore a sinusoid of amplitude 1 and frequency ω gets mapped through the LTI system to a sinusoid withe amplitude $M(\omega)$ and phase shift $\phi(\omega)$, where $M(\omega)$ and $\phi(\omega)$ are the magnitude and phase of the complex number $G(j\omega)$. So $G(j\omega)$ tells us everything we need to know about how a system responds to sinusoids.

$$\boxed{G(j\omega) \text{ is called the } \underline{\text{frequency response}} \text{ of the system.}}$$

Some intuition... Remember $\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt.$

Remember that $G(s) = \mathcal{L}\{g(t)\}$ where g is the impulse response.

The frequency response is: $G(j\omega) = \int_0^\infty e^{-j\omega t} g(t)\,dt = \int_{-\infty}^{\infty} e^{-j\omega t} g(t)\,dt$

since $g(t) = 0$ for $t < 0$.

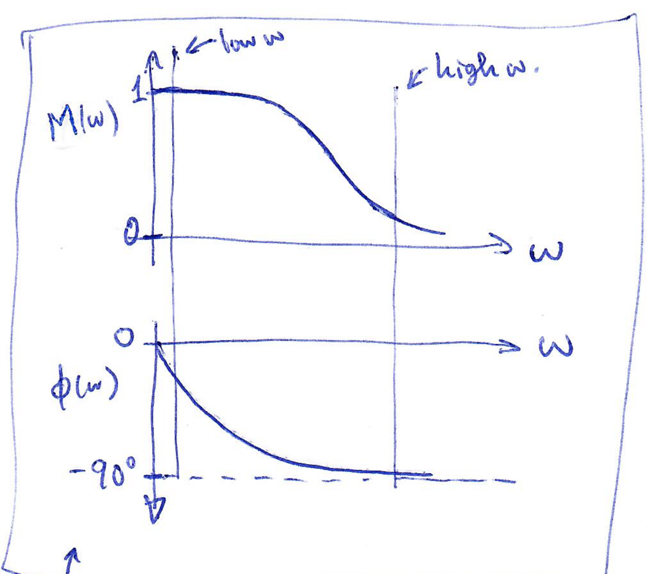
So $G(j\omega)$ is in fact the Fourier transform of the impulse response!

Example: Consider the 1st order system $G(s) = \dfrac{1}{s+1}$.

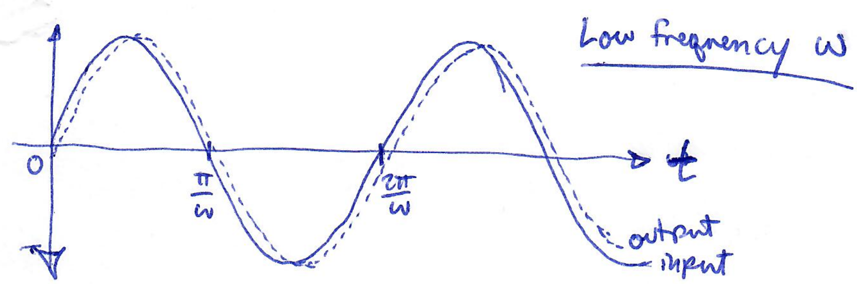
$$G(j\omega) = \frac{1}{1+j\omega} = \frac{1-j\omega}{(1+j\omega)(1-j\omega)} = \frac{1-j\omega}{1+\omega^2} = \underbrace{\left(\frac{1}{1+\omega^2}\right)}_{p(\omega)} + j\underbrace{\left(\frac{-\omega}{1+\omega^2}\right)}_{q(\omega)}$$

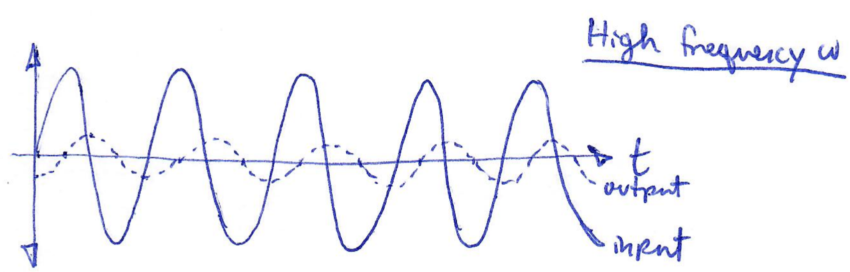
Therefore,

$$\boxed{\begin{aligned} M(\omega) &= \sqrt{p(\omega)^2 + q(\omega)^2} = \frac{1}{\sqrt{\omega^2+1}} \\ \text{and} \quad \phi(\omega) &= \arctan(-\omega) = -\arctan(\omega) \end{aligned}}$$



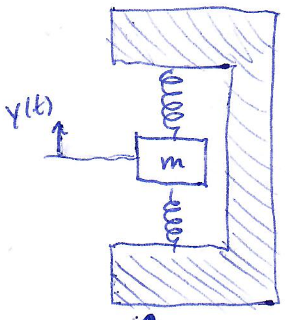
this is called a Bode plot
(pronounced Boh-dee)





NOTE: I plotted steady-state signals above!
(after transients have died out.)

Example: interactive demo:



accelerometer measures position of center mass (y) which is an indicator of the external force (u).

We can observe three distinct frequency regimes:

1) "low" frequency: if ω is small, $M(\omega) \approx 1$ and $\phi(\omega) \simeq 0°$ so input and output are roughly the same; the center mass tracks the force perfectly.

2) "medium" frequency: Here, $M(\omega) > 1$ (amplification!) and $\phi(\omega) \approx -90°$, so input and output are about $90°$ out of phase and output has larger amplitude than input.

3) "high" frequency: Here, $M(\omega) \approx 0$ (damping) and $\phi(\omega) \approx -180°$, so input and output are $180°$ out of phase and output has very small amplitude (does not track input at all).